Small slopes of Newton polygon of L-function

Fusheng Leng Banghe Li
Academy of Mathematics and Systems Science, Academia Sinica, China
KLMM

Abstract

To understand L-function is an important fundamental question in Number Theory, but there are few specific results on it, especially the calculation of its Newton polygon. Following Dwork's method it is hard to calculate an exact example, even on the case of one variable. There are only three such examples till now, one of which has some mistakes. In this paper we calculate L-functions with p-adic Gauss sums and give a formula in power series(**theorem 1.2.**). After that we discuss Newton polygons $\mathrm{NP}(f/\mathbf{F}_p, T)$ of L-functions of one variable polynomials and give a method to calculate its small slopes. We also obtain the Newton polygon $\mathrm{NP}(f/\mathbf{F}_q, T)$ of a 2-variables example with $f = x^3 + axy + by^2$ to illustrate our method.

1. **Introduction**

Let $\mathbf{F}_q$ be the finite field of q elements with characteristic p and $\mathbf{F}_{q^k}$ be the extension of $\mathbf{F}_q$ of degree k. Let ζ_p be a fixed primitive p-th root of unity in the complex numbers. For any Laurent polynomial $f(x_1, \ldots, x_n) \in \mathbf{F}_q[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, we form the exponential sum

$$S_k^*(f) = \sum_{x_i \in \mathbf{F}_{q^k}^*} \zeta_p^{\mathrm{Tr}_{\mathbf{F}_{q^k}/\mathbf{F}_p}(f(x_1,\cdots,x_n))}, \text{where } \mathbf{F}_{q^k}^* = \mathbf{F}_{q^k} \setminus \{0\}.$$

The L-function is defined by

$$L^*(f, T) = \exp(\sum_{k=1}^{\infty} S_k^*(f) \frac{T^k}{k}).$$

To understand the L-function is an important fundamental question in number theory, and since it is very difficult, there are only a few results on it.

By a theorem of Dwork-Bombieri-Grothendieck,

$$L^*(f, T) = \frac{\prod_{i=1}^{d_1}(1 - \alpha_i T)}{\prod_{j=1}^{d_2}(1 - \beta_j T)}$$

is a rational function, where the finitely many numbers α_i $(1 \le i \le d_1)$ and β_j $(1 \le j \le d_2)$ are non-zero algebraic integers. Equivalently, for each positive integer k, we have the formula

$$S_k^*(f) = \beta_1^k + \beta_2^k + \cdots + \beta_{d_2}^k - \alpha_1^k - \alpha_2^k - \cdots - \alpha_{d_1}^k.$$

Thus, our fundamental question about the sums $S_k^*(f)$ is reduced to understanding the reciprocal zeros α_i $(1 \le i \le d_1)$ and the reciprocal poles β_i $(1 \le j \le d_2)$. When we need to indicate the dependence of the L-function on the ground field $\mathbf{F}_q$, we will write $L^*(f/\mathbf{F}_q, T)$.

Without any smoothness condition on f, one does not even know exactly the number d_1 of zeros and the number d_2 of poles, although good upper bounds are available, see [4]. On the other hand, Deligne's theorem on the Riemann hypothesis [5] gives the following general information about the nature of the zeros and poles. For the complex absolute value $|\ |$, this says

$$|\alpha_i| = q^{\frac{u_i}{2}}, |\beta_j| = q^{\frac{v_j}{2}}, u_i \in \mathbf{Z} \cap [0, 2n], v_j \in \mathbf{Z} \cap [0, 2n]$$

where $\mathbf{Z} \cap [0, 2n]$ denotes the set of integers in the interval $[0, 2n]$. Furthermore, each α_i(resp. each β_j) and its Galois conjugates over $\mathbf{Q}$ have the same complex absolute value. For each l-adic absolute value $|\ |_l$ with prime $l \neq p$, the α_i and the β_j are l-adic units:

$$|\alpha_i|_l = |\beta_j|_l = 1.$$

For the remaining prime p, it is easy to prove

$$|\alpha_i|_p = q^{-r_i}, |\beta_j|_p = q^{-s_j}, r_i \in \mathbf{Q} \cap [0, n], s_j \in \mathbf{Q} \cap [0, n].$$

where we have normalized the p-adic absolute value by $|q|_p = q^{-1}$. Deligne's integrality theorem implies the following improved information:

$$r_i \in \mathbf{Q} \cap [0, n], s_j \in \mathbf{Q} \cap [0, n].$$

Our fundamental question is then to determine the important arithmetic invariants $\{u_i, v_j, r_i, s_j\}$.

Suppose

$$f = \sum_{j=1}^{J} a_j x^{V_j}, a_j \neq 0,$$

where each $V_j = (v_{1j}, \cdots, v_{nj})$ is a lattice point in $\mathbf{Z}^n$ and the power x^{V_j} simply means the product $x_1^{v_{1j}} \cdots x_n^{v_{nj}}$. Let $\Delta(f)$ be the convex closure in $\mathbf{R}^n$ generated by the origin and the lattice points V_j $(1 \leq j \leq J)$.

Definition 1.1. *The Laurent polynomial f is called non-degenerate if for each closed face δ of $\Delta(f)$ of arbitrary dimension which does not contain the origin, the n partial derivatives*

$$\{\frac{\partial f^\delta}{\partial x_1}, \cdots, \frac{\partial f^\delta}{\partial x_n}\}$$

have no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\boldsymbol{F}_q$.

If f is non-degenerate, the L-function $L^*(f/\mathbf{F}_q, T)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathbf{V}(f)$ by a theorem of Adolphson-Sperber, where $\mathbf{V}(f)$ denotes the volume of $\Delta(f)$. [2]

We are then interested in the Newton polygon of L-function. Dwork gave a method of cohomology theory with p-adic on determining Newton polygon in 1962 and 1964[26][27], and after that A.Adolphson and S.Sperber developed this method[2]. All these works depend on Dwork's trace formula

$$S_k^*(f) = (q^k - 1)^n Tr(\varphi^k)$$

and the definition of φ is given by lifting $\mathbf{F}_q$ to $\mathbf{Q}_q$ via a splitting function

$$\theta(t) = \sum_{m=0}^{\infty} \gamma_m t^m$$

where φ is an endomorphism of some p-adic Banach space.

However, there are still few general examples given by the method, especially when f is a polynomial with one variable. S.Sperber in 1986 gave the Newton polygon of $L^*(f,T)$ when $\deg f = 3$[16] and fifteen years later S.Hong gave two other examples with $\deg f = 4$ and $\deg f = 6$[9][10], which still had some mistakes in the last case. $\deg f = 5$ is more difficult to calculate than $\deg f = 6$ in two cases which need to prove some identical equations by hypergeometric summation theory, we will show that in our paper.

In 2003, R.Yang calculated the Newton polygon of $L(f/\mathbf{F}_p, T)$ on a special case when $f = x^d + \lambda x$ and $p \equiv -1 \mod d$ for p large enough[22].

In 2004, D.Wan gave a formula of L-function with Gauss sum in a special case when f is diagonal.[28]

A Laurent polynomial f is called **diagonal** if f has exactly n non-constant terms and $\Delta(f)$ is n-dimensional (necessarily a simplex), then we can write $f(x)$ as

$$f(x) = \sum_{i=1}^{n} a_i x^{V_i}, a_i \in \mathbf{F}_q^*.$$

Let S_p be the set of 0 and all rational numbers $a \in (0,1)$ such that $\mathrm{ord}_p a \geq 0$. For $a, b \in S_p$ define $a + b = c \in S_p$ where c is equal to the normal sum $a + b \mod 1$. It is not difficult to prove that $(S_p, +)$ isomorphs to $(\overline{\mathbf{F}}_p, \cdot)$ where $\overline{\mathbf{F}}_p$ is the algebraic closure of $\mathbf{F}_p$.

Consider the solutions of the following equation

$$(V_1, \ldots, V_n)(r_1, \ldots, r_n)^{\mathrm{T}} \equiv 0 \mod 1, r_i \text{ rational}, 0 \leq r_i < 1. \tag{1}$$

Let $S_p(\Delta)$ be the set of solutions r of equation above, such that $\mathrm{ord}_p r_i \geq 0$ for every $1 \leq i \leq n$, then $S_p(\Delta) \subset (S_p)^n$. Let $S_p(q,d)$ be the set of such $r \in S_p(\Delta)$ that $(q^d - 1)r \in \mathbf{Z}^n$ and $(q^{d'} - 1)r \notin \mathbf{Z}^n$ for every $1 \leq d' < d$. We have obviously the decomposition

$$S_p(\Delta) = \bigcup_{d \geq 1} S_p(q,d).$$

Let χ be the Teichm$\ddot{u}$ller character of the multiplicative group $\mathbf{F}_q^*$. Define Gauss sums over $\mathbf{F}_q$ by

$$G_k(q) = -\sum_{a \in \mathbf{F}_q^*} \chi(a)^{-k} \zeta_p^{\mathrm{Tr}(a)} (0 \leq k \leq q-2),$$

D.Wan has proved the following formula when the function f is diagonal

$$L^*(f/\mathbf{F}_q, T)^{(-1)^{n-1}} = \prod_{d \geq 1} \prod_{r \in S_p(q,d)} (1 - T^d \prod_{i=1}^{n} \chi(a_i)^{r_i(q^d-1)} G_{r_i(q^d-1)}(q^d))^{\frac{1}{d}}.$$

Note that (1) has finite number solutions since f is diagonal, and for each of the d points of $r \in S_p(q,d)$ the corresponding factors in this formula are the same. Thus it can be regarded as a polynomial.

The difficulty on improving this formula to general is, when f is not diagonal, there will be infinite factors in the formula.

It does not use the "diagonal" condition in the proof of the formula above and this condition only acts on whether the number of factors are finite. Note that $S_p(q,d)$ is a finite set, does not depend on whether the Laurent polynomial f is diagonal or not. So we can get a similar formula in general case, that is

Theorem 1.2. *Let $f(x) = \sum_{i=1}^m a_i x^{V_i} \in \boldsymbol{F}_q[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, where $a_i \neq 0$ for each i, and suppose $m > n$, then we have*

$$L^*(f/\boldsymbol{F}_q, T)^{(-1)^{n-1}} = \prod_{d\geq 1} \prod_{r\in S_p(q,d)} \prod_{h=0}^{\infty} (1 - q^{dh} T^d \prod_{i=1}^{m} \chi(a_i)^{(q^d-1)r_i} G_{(q^d-1)r_i}(q^d))^{\frac{C_{h+m-n-1}^{m-n-1}}{d}} \tag{2}$$

Remark 1.3. *if $r' \equiv q^s r \mod 1$ for $r, r' \in S_p(q,d)$ and for some integer s, the corresponding factors of r and r' in (2) are the same. Thus, we can remove the power $\frac{1}{d}$ if we restrict r to run over the q-orbits of $S_p(q,d)$. Because $S_p(q,d)$ is a finite set, we can easily prove that the right side of (2) is indeed a power series over $\boldsymbol{Z}_p(\pi)$, where $\boldsymbol{Z}_p$ is the ring of p-adic integers and π be the unique $(p-1)$-st root of $-p$ satisfying $\pi \equiv (\zeta_p - 1) \mod (\zeta_p - 1)^2$.*

Theorem 1.2. is not the rational function form of L-function, but we will show in this paper that this theorem is useful on the Newton polygon determination.

Denote the series in **remark 1.3.** by

$$L^*(f/\mathbf{F}_q, T)^{(-1)^{n-1}} = \sum_{s=0}^{\infty} c_s T^s.$$

The main idea in this paper is, to determine the Newton polygon by calculating $\mathrm{ord}_p c_s$ for every index s.

As we have known that

$$L(f,T) = \exp(\sum_{k=1}^{\infty} S_k(f) \frac{T^k}{k}),$$

where

$$S_k(f) = \sum_{x_i \in \mathbf{F}_{q^k}} \zeta_p^{\mathrm{Tr}_{\mathbf{F}_{q^k}/\mathbf{F}_p}(f(x_1,\cdots,x_n))},$$

one will see that $L^*(f,T) = (1-T)L(f,T)$ when $f(x)$ is a polynomial with one variable. Besides, for any $a_0 \in \mathbf{F}_q$, one can easily conclude that $L((f+a_0),T) = L(f, \zeta_p^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(a_0)} T)$. Thus we have $\mathrm{NP}((f+a_0)/\mathbf{F}_q, T) = \mathrm{NP}(f/\mathbf{F}_q, T)$. We can also easily conclude that such a linear transformation $x+b$ ($b \in \mathbf{F}_q$) of x does not change the L-function, this conclusion will be used to transform f to reduce our calculation of its Newton polygon.

In the last of this paper, we will give a new method to calculate Newton polygons of L-functions of one variable polynomials for first s slopes with $(s-2)(s-1) < 2d$ and for every p except some small values. All of these are based on **proposition 3.5.** and **theorem 6.1.**.

2. General theory

Let $f(x) = \sum_{i=1}^{m} a_i x^{V_i} \in \mathbf{F}_q[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, where $a_i \neq 0$ for each i, and suppose $m > n$.

To get the generalization of Wan's formula, we should describe $S_p(q,d)$ first.

Suppose that

$$k_1V_1 + \cdots + k_mV_m \equiv 0 \bmod (q^k - 1) \tag{3}$$

for a given positive integer k and $0 \leq k_i \leq q^k - 2$ for $i = 1, \cdots, m$.

The equation is equivalent to

$$\frac{k_1}{q^k-1}V_1 + \cdots + \frac{k_m}{q^k-1}V_m \equiv 0 \bmod 1 \tag{4}$$

Consider the equation

$$r_1V_1 + \cdots + r_mV_m \equiv 0 \bmod 1 \tag{5}$$

where $r_i \in \mathbf{Q}$, $0 \leq r_i < 1$ and, if $r_i = \frac{p_i}{q_i}$ with $(p_i, q_i) = 1$, then $(p, q_i) = 1$.

Define $S_p(f)$ the solution set of (5). It is clear that if (k_i) is a solution of (3), then $(\frac{k_i}{q^k-1}) \in S_p(f)$. Conversely, if $(r_i) \in S_p(f)$, assume $r_i = \frac{p_i}{q_i}$ with $(p_i, q_i) = 1$, we will show that (r_i) can be written as the form $(\frac{k_i}{q^k-1})$ for some positive integer k.

Since $(p, q_i) = 1$, following the Euler theorem in congruence theory we have

$$q^{\lambda_i} - 1 \equiv 0 \bmod q_i$$

where $\varphi(q_i)$ is the Euler function and $\lambda_i | \varphi(q_i)$ is the smallest positive integer x which satisfies $q^x - 1 \equiv 0 \bmod q_i$.

Let λ be the least common multiple of $\lambda_1, \cdots, \lambda_m$. Then

$$q^{\lambda} - 1 \equiv 0 \bmod q_i$$

for all i from 1 to m. Thus $r_i = \frac{p_i}{q_i}$ can be rewritten as the form $\frac{k_i}{q^{\lambda}-1}$. That is what we need.

Let $H_p(q,d)$ be the subgroup of $S_p(f)$ consisting of all such

$$r = (\frac{k_1}{q^d-1}, \cdots, \frac{k_m}{q^d-1})$$

with $0 \leq k_i \leq q^d - 2$. Then $H_p(q,d) \subset H_p(q,d')$ if $d|d'$ since $(q^d-1)|(q^{d'}-1)$.

Furthermore,

$$S_p(f) = \bigcup_{d \geq 1} H_p(q,d).$$

Define an action

$$q : r \to qr = (qr_1, \cdots, qr_m) \bmod 1$$

on $S_p(f)$. Let d be the number of the elements in the orbit of r under the action q. Then $r \in H_p(q,d)$ but r is not in any $H_p(q,d')$ for $d' < d$.

Let

$$S_p(q,d) = H_p(q,d) - \bigcup_{d'<d} H_p(q,d),$$

then

$$H_p(q,d) = \bigcup_{d'|d} S_p(q,d')$$

and

$$S_p(f) = \bigcup_{d\geq 1} S_p(q,d).$$

It is clear that every subset $S_p(q,d)$ is a finite set. In fact $|S_p(q,d)| \leq (q^d-1)^m$. Furthermore, assume the unique factorization $d = \prod_i p_i^{\alpha_i}$, following the principle of inclusion and exclusion we can also prove that $|S_p(q,d)| = (q^d-1) - \sum_i (q^{\frac{d}{p_i}} - 1) + \sum_{i\neq j}(q^{\frac{d}{p_i p_j}} - 1) - \cdots$.

Define Gauss sums over $\mathbf{F}_q$ by

$$G_k(q) = -\sum_{a\in \mathbf{F}_q^*} \chi(a)^{-k} \zeta_p^{\mathrm{Tr}(a)} (0 \leq k \leq q-2).$$

For each $a \in \mathbf{F}_q^*$, the Gauss sums satisfies the following interpolation relation

$$\zeta_p^{\mathrm{Tr}(a)} = \sum_{k=0}^{q-2} \frac{G_k(q)}{1-q} \chi(a)^k.$$

To get the generalization of Wan's formula, we also need a formula on Gauss sums. That is

Theorem 2.1. *(Hasse-Davenport) For every positive integer k,*

$$G_{r(q^{dk}-1)}(q^{dk}) = G_{r(q^d-1)}(q^d)^k.$$

Then we can calculate that

$$\begin{aligned}
S_1^*(f) &= \sum_{x_j\in \mathbf{F}_q^*} \zeta_p^{\mathrm{Tr}(f(x))} \\
&= \sum_{x_j\in \mathbf{F}_q^*} \prod_{i=1}^{m} \zeta_p^{\mathrm{Tr}(a_i x^{V_i})} \\
&= \sum_{x_j\in \mathbf{F}_q^*} \prod_{i=1}^{m} \sum_{k_i=0}^{q-2} \frac{G_{k_i}(q)}{1-q} \chi(a_i)^{k_i} \chi(x^{V_i})^{k_i} \\
&= \sum_{k_1=0}^{q-2} \cdots \sum_{k_m=0}^{q-2} \Big(\prod_{i=1}^{m} \frac{G_{k_i}(q)}{1-q} \chi(a_i)^{k_i}\Big) \sum_{x_j\in \mathbf{F}_q^*} \chi(x^{k_1V_1+\cdots+k_mV_m}).
\end{aligned}$$

Note that $\sum_{x_j\in\mathbf{F}_q^*}\chi(x^{k_1V_1+\cdots+k_mV_m})=0$ unless $k_1V_1+\cdots+k_mV_m\equiv 0 \mod q-1$, if this condition holds, the value of $\sum_{x_j\in\mathbf{F}_q^*}\chi(x^{k_1V_1+\cdots+k_mV_m})$ will be $(q-1)^n$. Thus,

$$S_1^*(f)=(-1)^n(1-q)^{n-m}\sum_{k_1V_1+\cdots+k_mV_m\equiv 0 \mod q-1}\prod_{i=1}^m\chi(a_i)^{k_i}G_{k_i}(q).$$

Replacing q by q^k, one gets a formula for the exponential sum $S_k^*(f)$ over the k-th extension field $\mathbf{F}_{q^k}$:

$$\begin{aligned}S_k^*(f)&=(-1)^n(1-q^k)^{n-m}\sum_{r\in H_p(q,k)}\prod_{i=1}^m\chi(a_i)^{r_i(q^k-1)}G_{r_i(q^k-1)}(q^k)\\ &=\sum_{k'|k}\sum_{r\in S_p(q,k')}(-1)^n(1-q^k)^{n-m}\prod_{i=1}^m\chi(a_i)^{r_i(q^k-1)}G_{r_i(q^k-1)}(q^k).\end{aligned}\tag{6}$$

Since

$$L^*(f/\mathbf{F}_q,T)=\exp(\sum_{k=1}^\infty S_k^*(f)\frac{T^k}{k}),$$

by (6) the equation above is equal to

$$\prod_{d\ge 1}\prod_{r\in S_p(q,d)}\exp(\sum_{k=1}^\infty\frac{T^{kd}}{kd}(-1)^n(1-q^{kd})^{n-m}\prod_{i=1}^m\chi(a_i)^{r_i(q^{kd}-1)}G_{r_i(q^{kd}-1)}(q^{kd})).$$

Following the Hasse-Davenport relation we rewrite it by

$$\begin{aligned}L^*(f/\mathbf{F}_q,T)&=\prod_{d\ge 1}\prod_{r\in S_p(q,d)}\exp(\sum_{k=1}^\infty\frac{T^{kd}}{kd}(-1)^n(1-q^{kd})^{n-m}\prod_{i=1}^m\chi(a_i)^{kr_i(q^d-1)}G_{r_i(q^d-1)}(q^d)^k)\\ &=\prod_{d\ge 1}\prod_{r\in S_p(q,d)}\exp(\sum_{k=1}^\infty\frac{T^{kd}}{kd}(-1)^n\sum_{h=0}^\infty C_{h+m-n-1}^{m-n-1}q^{kdh}\prod_{i=1}^m\chi(a_i)^{kr_i(q^d-1)}G_{r_i(q^d-1)}(q^d)^k)\\ &=\prod_{d\ge 1}\prod_{r\in S_p(q,d)}\prod_{h=0}^\infty\exp(\sum_{k=1}^\infty\frac{T^{kd}}{kd}(-1)^nC_{h+m-n-1}^{m-n-1}q^{kdh}\prod_{i=1}^m\chi(a_i)^{kr_i(q^d-1)}G_{r_i(q^d-1)}(q^d)^k)\\ &=(\prod_{d\ge 1}\prod_{r\in S_p(q,d)}\prod_{h=0}^\infty(1-q^{dh}T^d\prod_{i=1}^m\chi(a_i)^{r_i(q^d-1)}G_{r_i(q^d-1)}(q^d))^{\frac{C_{h+m-n-1}^{m-n-1}}{d}})^{(-1)^{n-1}}.\end{aligned}$$

This is the proof of **Theorem 1.2.**.

We denote the $h=0$-part in (2) as

$$L_0^*(f/\mathbf{F}_q,T)=\prod_{d\ge 1}\prod_{r\in S_p(q,d)}(1-T^d\prod_{i=1}^m\chi(a_i)^{(q^d-1)r_i}G_{(q^d-1)r_i}(q^d))^{\frac{1}{d}}.$$

Recall that $S_1^*(f)=(-1)^n(1-q)^{n-m}\sum_{k_1V_1+\cdots+k_mV_m\equiv 0 \mod q-1}\prod_{i=1}^m\chi(a_i)^{k_i}G_{k_i}(q)$ the $h=0$-part in (2) is indeed the $\exp(\sum_{k=1}^\infty S_k^*(f)\frac{T^k}{k})$ replacing $S_k^*(f)$ by $(1-q^k)^{m-n}S_k^*(f)$.

Theorem 2.2. *Suppose f is non-degenerate, then the Newton polygon of $L^*(f/\boldsymbol{F}_q, T)$ is the same as $L_0^*(f/\boldsymbol{F}_q, T)$ up to the slopes smaller than 1. Especially, if $f(x) = \sum_{i=1}^m a_i x^{d_i}$, where $0 < d_1 < \cdots < d_m = d$, $a_i \in \boldsymbol{F}_q^*$ and $d \neq 0$ mod p, then f is non-degenerate and the Newton polygon of $L^*(f/\boldsymbol{F}_q, T)$ is the same as $L_0^*(f/\boldsymbol{F}_q, T)$ up to d.*

Proof. Since

$$L_0^*(f/\mathbf{F}_q, T) = \exp(\sum_{k=1}^{\infty}(1-q^k)^{m-n} S_k^*(f)\frac{T^k}{k}),$$

we have

$$\begin{aligned} L_0^*(f/\mathbf{F}_q, T) &= \exp(\sum_{k=1}^{\infty}\sum_{i=0}^{m-n} C_{m-n}^i (-1)^i q^{ki}\frac{S_k^*(f)T^k}{k}) \\ &= \prod_{i=0}^{m-n}(\exp(\sum_{k=1}^{\infty}\frac{S_k^*(f)(Tq^i)^k}{k}))^{C_{m-n}^i(-1)^i} \\ &= \frac{L^*(f/\mathbf{F}_q, T)L^*(f/\mathbf{F}_q, q^2T)^{C_{m-n}^2}\cdots}{L^*(f/\mathbf{F}_q, qT)^{m-n}L^*(f/\mathbf{F}_q, q^3T)^{C_{m-n}^3}\cdots}. \end{aligned}$$

Recall that

$$L^*(f, T) = \frac{\prod_{i=1}^{d_1}(1-\alpha_i T)}{\prod_{j=1}^{d_2}(1-\beta_j T)}$$

where

$$\mathrm{ord}_q\alpha_i = r_i, \mathrm{ord}_q\beta_j = s_j, r_i \in \mathbf{Q}\cap[0,n], s_j \in \mathbf{Q}\cap[0,n].$$

with normalized the p-adic order by $\mathrm{ord}_q q = 1$, then the reciprocal zeros $q^k\alpha_i$ $(1 \le i \le d_1)$ and the reciprocal poles $q^k\beta_i$ $(1 \le j \le d_2)$ of $L^*(f/\mathbf{F}_q, q^kT)$ have the p-adic orders

$$\mathrm{ord}_q(q^k\alpha_i) = r_i + k, \mathrm{ord}_q(q^k\beta_j) = s_j + k, r_i \in \mathbf{Q}\cap[0,n], s_j \in \mathbf{Q}\cap[0,n].$$

Thus, the only reciprocal zeros or reciprocal poles in $L_0^*(f/\mathbf{F}_q, T)$ for which the p-adic orders smaller than 1 must appear in $L^*(f/\mathbf{F}_q, T)$ of the right side of the equation above.

Then we obtain the theorem. □

Following **theorem 2.2.**, to calculate the Newton polygon of $L^*(f/\mathbf{F}_q, T)$ where $f(x) = \sum_{i=1}^m a_i x^{d_i}$ we should only determine the ord_q-value of $L_0^*(f/\mathbf{F}_q, T)$'s first d-terms.

3. Presentation of $L_0^*(f/\mathbf{F}_q, T)$'s term

To express clearly, we first give some symbols that will be used below.

Definition 3.1. *For an arbitrary real number x, define $\{x\}$ satisfying that $0 \leq \{x\} < 1$ and $\{x\} \equiv x$ mod 1.*

Definition 3.2. *Any nonnegative integer k can be written as the form*

$$k = k_0 + pk_1 + \cdots + p^{l-1}k_{l-1}$$

uniquely, where k_t is an integer and $0 \leq k_t \leq p-1$ for each $0 \leq t < l$ and $k_{l-1} \neq 0$. Define a function σ on the nonnegative integer set to itself such that

$$\sigma(k) = \sum_{t=0}^{l-1} k_t.$$

Following these symbols we can express the Gross-Koblitz formula as below:

Theorem 3.3. *(Gross-Koblitz) Suppose $p \geq 2$ prime and $q = p^a$. Let π be the unique $(p-1)$-st root of $-p$ satisfying*

$$\pi \equiv (\zeta_p - 1) \ mod \ (\zeta_p - 1)^2.$$

Then

$$G_k(q) = \pi^{\sigma(k)} \Pi_{j=0}^{a-1} \Gamma_p(\{\frac{p^j k}{q-1}\}).$$

Let $f(x) = \sum_{i=1}^m a_i x^{d_i}$, where $0 < d_1 < \cdots < d_m = d$, $\gcd(p,d) = 1$, $a_i \in \mathbf{F}_q^*$. Thus f is non-degenerate and $L^*(f/\mathbf{F}_q, T)$ is a polynomial of degree d.

Recall the definition of $L_0^*(f/\mathbf{F}_p, T)$ we denote $L_0^*(f/\mathbf{F}_p, T) = \sum_{s=0}^{\infty} c_s T^s$. For a given integer s, the c_s can be expressed as the sum of all these terms:

$$\prod_{\sum s_j = s, r_j \in S_p(p, s_j)} (-\prod_{i=1}^m \chi(a_i)^{(p^{s_j}-1)r_{ji}} G_{(p^{s_j}-1)r_{ji}}(p^{s_j})). \tag{7}$$

Following Gross-Koblitz formula (7) can be written as

$$\prod_{\sum s_j = s, r_j \in S_p(p, s_j)} (-\prod_{i=1}^m \chi(a_i)^{(p^{s_j}-1)r_{ji}} G_{(p^{s_j}-1)r_{ji}}(p^{s_j}))$$

$$= \prod_{i=1}^m (\chi(a_i)^{\sum_j \sigma((p^{s_j}-1)r_{ji})} \pi^{\sum_j \sigma((p^{s_j}-1)r_{ji})}) \cdot \prod_j (-\prod_{i=1}^m \prod_{t=0}^{s_j-1} \Gamma_p(\{p^t r_{ji}\})). \tag{8}$$

There are infinite number of such addends, but only finite number of them have the ord_p-value smaller than a given number.

Consider (8), let

$$k_{ji} = (p^{s_j} - 1) r_{ji} = \sum_{t=0}^{s_j-1} k_{ji}[t] p^t$$

and

$$\sum_{i=1}^{m} d_i k_{ji}[t] = u_j[t]p - v_j[t]$$

where $0 \le k_{ji}[t] \le p-1$ and $0 \le v_j[t] \le p-1$ and $0 \le u_j[t] \le \sum_{i=1}^{m} d_i$.

Proposition 3.4. *Suppose $p \ge \sum_{i=1}^{m} d_i$. For any index j,*

$$u_j[t-1] = v_j[t] \tag{9}$$

for every $0 \le t \le s_j - 1$.

Conversely, let

$$k_{ji} = \sum_{t=0}^{s_j-1} k_{ji}[t]p^t$$

and

$$\sum_{i=1}^{m} d_i k_{ji}[t] = u_j[t]p - v_j[t]$$

for $0 \le k_{ji}[t] \le p-1$ and $0 \le v_j[t] \le p-1$ and $0 \le u_j[t] \le \sum_{i=1}^{m} d_i$. if the condition

$$u_j[t-1] = v_j[t]$$

for every $0 \le t \le s_j - 1$ is achieved, then

$$r_j = (\frac{k_{j1}}{p^{s_j}-1}, \cdots, \frac{k_{jm}}{p^{s_j}-1}) \in S_p(p, s_j).$$

Proof. Since $r_j \in S_p(p, s_j)$ we have

$$\sum_{i=1}^{m} d_i k_{ji} \equiv 0 \bmod p^{s_j} - 1.$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{m} d_i k_{ji} &= \sum_{i=1}^{m} \sum_{t=0}^{s_j-1} d_i k_{ji}[t]p^t \\
&= \sum_{t=0}^{s_j-1} \sum_{i=1}^{m} d_i k_{ji}[t]p^t \\
&= \sum_{t=0}^{s_j-1} (u_j[t]p - v_j[t])p^t \\
&= u_j[s_j-1](p^{s_j}-1) + \sum_{t=0}^{s_j-1} (u_j[t-1] - v_j[t])p^t,
\end{aligned}$$

where $u_j[-1] = u_j[s_j - 1]$.

Then

$$\sum_{i=1}^{m} d_i k_{ji} \equiv 0 \bmod p^{s_j} - 1$$

is equivalent to

$$\sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t \equiv 0 \bmod p^{s_j}-1. \tag{10}$$

Recall that

$$\sum_{i=1}^{m} d_i k_{ji}[t] = u_j[t]p - v_j[t],$$

by $p \geq \sum_{i=1}^{m} d_i$, $0 \leq k_{ji}[t] \leq p-1$ and $0 \leq v_j[t] \leq p-1$ we have

$$\begin{aligned} u_j[t] &= \frac{1}{p}(\sum_{i=1}^{m} d_i k_{ji}[t] + v_j[t]) \\ &\leq \frac{1}{p}(\sum_{i=1}^{m} d_i k_{ji}[t] + (p-1)) \\ &\leq \frac{1}{p}(1+\sum_{i=1}^{m} d_i)(p-1) \\ &\leq \frac{1}{p}(1+p)(p-1) \\ &< p \end{aligned}$$

i.e.

$$u_j[t] \leq p-1.$$

Then we have

$$\begin{aligned} \sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t &\leq \sum_{t=0}^{s_j-1} u_j[t-1]p^t \\ &\leq \sum_{t=0}^{s_j-1}(p-1)p^t \\ &= p^{s_j}-1 \end{aligned}$$

If the equality is achieved, it obtains that

$$v_j[t]=0, u_j[t]=p-1 \tag{11}$$

for every $0 < t \leq s_j - 1$. Furthermore, by $\sum_{i=1}^{m} d_i k_{ji}[t] = u_j[t]p - v_j[t]$ and $0 \leq k_{ji}[t] \leq p-1$, (11) means that

$$p = \sum_{i=1}^{m} d_i$$

and

$$k_{ji}[t] = p-1$$

for every $0 < t \leq s_j - 1$ and $1 \leq i \leq m$, but it is contradictory to $k_{ji} < p^{s_j} - 1$.

Thus

$$\sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t < p^{s_j}-1. \tag{12}$$

Besides,

$$\begin{aligned}\sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t &\geq -\sum_{t=0}^{s_j-1} v_j[t]p^t \\ &\geq -\sum_{t=0}^{s_j-1}(p-1)p^t \\ &= -(p^{s_j}-1)\end{aligned}$$

The equality can not be achieved since $u_j[t]p - v_j[t] = \sum_{i=1}^m d_i k_{ji}[t] \geq 0$ for every $0 < t \leq s_j - 1$, i.e.

$$-(p^{s_j}-1) < \sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t. \tag{13}$$

Following (10), (12) and (13) we obtain that

$$\sum_{t=0}^{s_j-1}(u_j[t-1]-v_j[t])p^t = 0,$$

i.e.

$$u_j[t-1] = v_j[t]$$

for every $0 \leq t \leq s_j - 1$.

To prove the remainder of this Proposition, we should only show that

$$\sum_{i=1}^{m} d_i k_{ji} \equiv 0 \bmod p^{s_j} - 1$$

when the condition is achieved. This is obvious. $\square$

Following the notations above we construct a table with many blocks as below:

$$\begin{pmatrix} k_{11}[0] & k_{11}[1] & \cdots & k_{11}[s_1-1] \\ k_{12}[0] & k_{12}[1] & \cdots & k_{12}[s_1-1] \\ \cdots & \cdots & \cdots & \cdots \\ k_{1m}[0] & k_{1m}[1] & \cdots & k_{1m}[s_1-1] \end{pmatrix} \cdots \begin{pmatrix} k_{j1}[0] & k_{j1}[1] & \cdots & k_{j1}[s_j-1] \\ k_{j2}[0] & k_{j2}[1] & \cdots & k_{j2}[s_j-1] \\ \cdots & \cdots & \cdots & \cdots \\ k_{jm}[0] & k_{jm}[1] & \cdots & k_{jm}[s_j-1] \end{pmatrix} \cdots$$

where $\sum_j s_j = s$.

A block

$$\begin{pmatrix} k_{j1}[0] & k_{j1}[1] & \cdots & k_{j1}[s_j-1] \\ k_{j2}[0] & k_{j2}[1] & \cdots & k_{j2}[s_j-1] \\ \cdots & \cdots & \cdots & \cdots \\ k_{jm}[0] & k_{jm}[1] & \cdots & k_{jm}[s_j-1] \end{pmatrix}$$

in the table is corresponding to such a factor

$$-\prod_{i=1}^{m}\chi(a_i)^{(p^{s_j}-1)r_{ji}}G_{(p^{s_j}-1)r_{ji}}(p^{s_j})$$

in (7). This table is determined by the c_s' term which we have chosen.

Conversely, if such a table is given, satisfying the relation in **Proposition 3.4.**, each block is corresponding to a $r_j \in S_p(p, s_j)$, where s_j is the number of the block's columns, and each of such r_j is in the different orbits, then the table determines a term of c_s.

Following (8) a term of c_s corresponding to such a table above has the ord_p-value equal to

$$\mathrm{ord}_p\pi \cdot (\sum_j \sigma((p^{s_j}-1)r_{ji})),$$

which is $\frac{1}{p-1}$ products the sum of all $k_{ji}[t]$ that appear in the table, i.e.

$$\frac{1}{p-1}\sum_j\sum_{i=1}^{m}\sum_{t=0}^{s_j-1}k_{ji}[t].$$

Let $T(c_s)$ be the set consisting of all the terms of c_s, and $I(c_s) = \{\mathrm{ord}_p c \mid c \in T(c_s)\}$. Rewrite $I(c_s)$ to $\{\alpha_1, \cdots, \alpha_k, \cdots\}$, where $\alpha_k < \alpha_{k'}$ when $k < k'$, and denote $T_k = \{c \in T(c_s) \mid \mathrm{ord}_p c = \alpha_k\}$. To calculate $\mathrm{ord}_p c_s$, we should calculate

$$\mathrm{ord}_p \sum_{c\in T_1} c.$$

If it is equal to α_1, then $\mathrm{ord}_p c_s$ is also equal to α_1, otherwise we will show in the next two sections that

$$\mathrm{ord}_p \sum_{c\in T_1} c \geq \alpha_1 + 1,$$

and we should calculate

$$\mathrm{ord}_p \sum_{c\in T_2} c,$$

and so on.

We will point out that the sum of all terms as form (8) whose ord_p-value smaller than a certain number has the ord_p-value greater than 1, that means we can submit those terms when calculating $\mathrm{ord}_p c_s$ and begin from some greater value of α_k. Indeed, we can get

Proposition 3.5. *Suppose $p \geq \sum_{i=1}^{m} d_i$. For $s > 0$, we have*

$$ord_p \sum c \geq 1 + \frac{s-1}{d}$$

for all such $c \in T(c_s)$ that, there are two same u-values $u_j[t] = u_{j'}[t']$ in the table corresponding to c.

Furthermore, if $(s-2)(s-1) < 2d$, to calculate $ord_p c_s$ we should only begin from $ord_p \sum c$ for all such $c \in T(c_s)$ that corresponds to s different u-values as $0, 1, \cdots, s-1$ respectively.

To prove this proposition, we will introduce some definitions and conclusions in the next two sections.

Denote $w_1, \dots, w_s$ the s column vectors of such $(k_{ji}[t])_{i=1,\dots,m}$. It is only a symbol in general that have no $k_{ji}[t]$'s values yet. When the values of $k_{ji}[t]$ are fixed, we denote s vectors $\overline{w_1}, \dots, \overline{w_s}$ as the values of $w_1, \dots, w_s$ respectively. Note that every w_j is different since it is just a symbol, and some of $\overline{w_j}$ may have the same value of u or v and, even they are all the same as vectors.

4. f-simple permutation on the symmetric group

In this section, f is a given map defined on the set $\{1, 2, 3, \dots, n\}$ and maps to an arbitrary set, and S_n is the symmetric group of $\{1, 2, 3, \dots, n\}$. Note that a permutation can be written as the product of several separated **cycles** uniquely.

It is easily to see that all $\sigma \in S_n$ satisfying

$$f(\sigma(i)) = f(i)$$

for every $i \in \{1, 2, 3, \dots, n\}$ form a subgroup of S_n, we name this subgroup by G_f. In fact a σ in G_f is a permutation among the inverse images of f respectively, so G_f isomorphs to the direct product of some symmetric groups.

For example, let $n = 7$, $f(1) = f(2) = f(3) = f(4) = \alpha$, $f(5) = f(6) = f(7) = \beta$, then $f^{-1}(\alpha) = \{1, 2, 3, 4\}$, $f^{-1}(\beta) = \{5, 6, 7\}$, and $G_f \simeq S_4 \times S_3$.

Definition 4.1. *For a given permutation $a \in S_n$, if the centralizer of a in G_f is $\{id\}$, we call the permutation a f-simple.*

For example, $f(i) = i$ for every $i \in \{1, 2, 3, \dots, n\}$, then $G_f = \{id\}$ and every permutation $a \in S_n$ is f-simple. Another extreme case is when f is a constant map, where $G_f = S_n$ and then only $id \in S_n$ is f-simple.

It is clear that if a $\sigma \neq id \in G_f$ satisfies $\sigma a = a\sigma$, then $f(a\sigma(i)) = f(\sigma a(i)) = f(a(i))$ for each i. Furthermore, $f(a^k\sigma(i)) = f(\sigma a^k(i)) = f(a^k(i))$ for each integer k.

For $\sigma \neq id$, there exists a number $i_0 \in \{1, 2, 3, \dots, n\}$ satisfying $\sigma(i_0) \neq i_0$.

Lemma 4.2. *For a given permutation $a \in S_n$ and $\sigma \in G_f$ is a centralizer element of a, suppose $\sigma(i_0) \neq i_0$ for a certain number $i_0 \in \{1, 2, 3, \dots, n\}$.*

If $\sigma(i_0) = a^d(i_0)$ for some $d > 0$, then there exist a integer $d_0 > 0$ and $k > 1$, such that the cycle of permutation a which including i_0 can be written as the form

$$(i_0 \cdots i_{d_0-1} i_{d_0} \cdots i_{kd_0-1})$$

where $f(i_{cd_0+t}) = f(i_t)$ for $t = 0, 1, \dots, d_0 - 1$ and $c = 0, 1, \dots, k - 1$.

Proof. Since $\sigma(i_0) = a^d(i_0)$ and σ is a centralizer element of a, we have

$$\sigma a(i_0) = a\sigma(i_0) = a^{d+1}(i_0)$$

$$\sigma a^2(i_0) = a^2\sigma(i_0) = a^{d+2}(i_0)$$

and so on. Following this and $\sigma \in G_f$ we obtain that

$$f(a^d(i_0)) = f(\sigma(i_0)) = f(i_0),$$
$$f(a^{d+1}(i_0)) = f(\sigma a(i_0)) = f(a(i_0))$$
$$\cdots\cdots$$
$$f(a^{d+j}(i_0)) = f(\sigma a^j(i_0)) = f(a^j(i_0))$$
$$\cdots\cdots$$

We then obtain that

$$f(a^{hd+j}(i_0)) = f(a^j(i_0))$$

for every non-negative integers h and j.

Assume $m = \min\{c > 0 \mid a^c(i_0) = i_0\}$, $d_0 = \gcd(m, d)$. Thus $d_0 < m$ since $a^d(i_0) = \sigma(i_0) \neq i_0$. Then we can rewrite this cycle of permutation a to

$$(i_0 \cdots i_{d_0-1} i_{d_0} \cdots i_{kd_0-1}),$$

where $k > 1$ and $m = kd_0$.

Since $a^m(i_0) = i_0$ we have

$$f(a^{hm+j}(i_0)) = f(a^j(i_0))$$

for every non-negative integers h and j. Furthermore, we can write

$$d_0 = um + vd$$

for some integers u and v since d_0 is the greatest common divisor of m and d. Following these we have

$$f(a^{hd_0+j}(i_0)) = f(a^{hum+hvd+j}(i_0)) = f(a^j(i_0))$$

for every non-negative integers h and j. Following this we finish our proof. □

Lemma 4.3. *For a given permutation $a \in S_n$ and $\sigma \in G_f$ is a centralizer element of a, suppose $\sigma(i_0) \neq i_0$ for a certain number $i_0 \in \{1, 2, 3, \ldots, n\}$.*

Denote the cycle of permutation a which including i_0 as

$$(i_0 \cdots i_{m-1}).$$

If $\sigma(i_0) \neq a^d(i_0)$ for every $d > 0$, then the cycle of permutation a that including $\sigma(i_0)$ has the form

$$(j_0 \cdots j_{m-1}),$$

where $j_0 = \sigma(i_0)$. Therefore

$$f(i_k) = f(j_k)$$

for every $0 \leq k \leq m-1$.

Proof. Let $j_k = a^k\sigma(i_0)$, $0 \leq k \leq m-1$. Then these m integers are all different with each other since $a\sigma = \sigma a$, i.e. $(j_0 \cdots j_{m-1})$ is a cycle of permutation a. Because $\sigma(i_0) \neq a^d(i_0)$ for every $d > 0$, it shows that the cycle $(j_0 \cdots j_{m-1})$ is different with $(i_0 \cdots i_{m-1})$. □

The inverse of **lemma 4.2.** and **lemma 4.3.** are also right, that is

Lemma 4.4. *Suppose $a \in S_n$ is a permutation of $\{1, 2, 3, \ldots, n\}$.*

(i) If there is a cycle of the permutation a has form

$$(i_0 \cdots i_{d-1} i_d \cdots i_{kd-1})$$

where $d \geq 1, k > 1$ and $f(i_{cd+t}) = f(i_t)$ for $t = 0, 1, \ldots, d-1$ and $c = 0, 1, \ldots, k-1$, then the permutation a is not f-simple.

(ii) If there are two cycles of the permutation a have form

$$(i_0 \cdots i_{d-1})$$

and

$$(j_0 \cdots j_{d-1})$$

such that $d \geq 1$ and

$$f(i_k) = f(j_k)$$

for every $0 \leq k \leq d-1$, then the permutation a is not f-simple.

Proof. For (i) the permutation

$$\sigma = (i_0 i_d i_{2d} \cdots i_{(k-1)d})(i_1 i_{d+1} i_{2d+1} \cdots i_{(k-1)d+1}) \cdots (i_{d-1} i_{2d-1} i_{3d-1} \cdots i_{kd-1})$$

satisfies that $\sigma a = a\sigma$ and $\sigma \neq id$.

For (ii) the permutation

$$\sigma = (i_0 j_0)(i_1 j_1) \cdots (i_{d-1} j_{d-1})$$

satisfies that $\sigma a = a\sigma$ and $\sigma \neq id$. $\square$

These lemmas give us an equivalent condition to the f-simple.

Each cycle in a given permutation can be written as the form

$$(i_0 \cdots i_{d-1} i_d \cdots i_{kd-1})$$

where $d \geq 1, k \geq 1$ and

$$f(i_{cd+t}) = f(i_t)$$

for $t = 0, 1, \ldots, d-1$ and $c = 0, 1, \ldots, k-1$. For example, let $k = 1$. We interest in how great k can achieve for the given cycle. If such a k is the greatest one satisfying the condition above, then the vectors

$$(f(i_0), f(i_1), \ldots, f(i_{d-1})),$$

$$(f(i_1), f(i_2), \ldots, f(i_{d-1}), f(i_0)),$$

$$\ldots,$$

$$(f(i_{d-1}), f(i_0), \ldots, f(i_{d-2}))$$

are all different with each other, we call the set that formed by these vectors the **f-kernel** of this given cycle. Note that f-kernel is uniquely determined by the cycle and the map f.

Proposition 4.5. *Suppose a set $G \subset S_n$ satisfying $\sigma G = G$ for all $\sigma \in G_f$, then in G the number of even no-f-simple permutations is equal to the number of odd no-f-simple permutations.*

Proof. Let $\tau \in G$, it can be written as the product of several separated cycles uniquely. We classify these cycles by their f-kernels and any class C has the form below:

$$C = (i_{10} \cdots i_{1(d-1)} i_{1d} \cdots i_{1(k_1 d-1)})(i_{20} \cdots i_{2(d-1)} i_{2d} \cdots i_{2(k_2 d-1)}) \cdots (i_{j0} \cdots i_{j(d-1)} i_{jd} \cdots i_{j(k_j d-1)}),$$

where $f(i_{s(cd+t)}) = f(i_{1t})$ for $t = 0, 1, \ldots, d-1$, $s = 1, \ldots, j$ and $c = 0, 1, \ldots, k_s - 1$, i.e. these cycles in C have same f-kernel.

Let σ_C be a permutation among $i_{s(cd+t)}$ which have same value of t respectively, then $\sigma_C \in G_f$. Since σ_C can be written as the product of several transpositions, it is easily to prove that all cycles in $\sigma_C C$ have the f-kernel same as of cycles in C.

Assume $\tau = C_1 \cdot \cdots \cdot C_m$ where C_j is the product of the cycles of τ which have the same f-kernel, $j = 1, \cdots, m$. Since $\sigma G = G$ for all $\sigma \in G_f$, we define the set $\{\prod_{j=1}^{m} \sigma_{C_j} C_j\}$ for any σ_{C_j} respect with C_j to be the equivalent class of τ over G. This equivalent relation over G is well-defined since all cycles in $\sigma_C C$ have the same f-kernel as of cycles in C.

Note that all σ_C form a subgroup of G_f which isomorphs to $(S_{\sum_{s=1}^{j} k_s})^d$ with $S_{\sum_{s=1}^{j} k_s}$ the symmetric group of degree $\sum_{s=1}^{j} k_s$, then following $G_f G = G$ the equivalent class of τ over G is τ left multiplied by such subgroup of G_f which isomorphs to the product of $(S_{\sum_{s=1}^{j} k_s})^d$. If τ is no-f-simple permutation, then some $\sum_{s=1}^{j} k_s > 1$ by **lemma 4.2.** and **lemma 4.3.**. Thus in the equivalent class of τ over G, the number of even no-f-simple permutations is equal to the number of odd no-f-simple permutations. This is what we want to prove. □

Assume a is an f-simple permutation, then for any $\sigma \in G_f$, the conjugation $\sigma a \sigma^{-1}$ is not equal to a when $\sigma \neq id$. Therefore $|\{\sigma a \sigma^{-1} \mid \sigma \in G_f\}| = |G_f|$, and following **proposition 4.5.** we have

Proposition 4.6. *Suppose a set $G \subset S_n$ satisfying $\sigma G = G$ and $\sigma a \sigma^{-1} \in G$ for all $\sigma \in G_f$ and all f-simple permutation $a \in G$. If the number of even permutations in G is equal to the number of odd permutations in G, then in G the number of such conjugate classes*

$$\{\sigma a \sigma^{-1} \mid \sigma \in G_f\}$$

where $a \in G$ is even f-simple permutation, is equal to the number of such conjugate classes where $a \in G$ is odd f-simple permutation.

5. **Permutation of w**

By the reason in **section 3**, a term as (7) determines such a table in **section 3**. Recalling the definition of $\overline{w_j}$, a term as (7) also determines a set $W = \{\overline{w_1}, \cdots, \overline{w_s}\}$ of m-dimension vectors(same elements do not combine). We say that two terms as (7) are equivalent, if they determine same set of W. Obviously this is an equivalent

relationship, and for each term in the same equivalent class, following equation (8) we will see that they have the same part of $\prod_{i=1}^{m}(\chi(a_i)^{\sum_j \sigma((p^{s_j}-1)r_{ji})}\pi^{\sum_j \sigma((p^{s_j}-1)r_{ji})})$ since $\sum_j \sigma((p^{s_j}-1)r_{ji})$ is equal to the sum of all $k_{ji}[t]$ that in the table.

Besides, following Stickelberger theorem the part

$$\prod_j(-\prod_{i=1}^{m}\prod_{t=0}^{s_j-1}\Gamma_p(\{p^t r_{ji}\}))$$

mod p is equal to $\prod_j \prod_{i=1}^{m}\prod_{t=0}^{s_j-1}\frac{1}{k_{ji}[t]!}$ or its inverse, it depends on whether the number of index j is even or not, i.e. it depends on whether the number of blocks in the corresponding table is even or not. This means all terms in the same equivalent class are at most different with a sign after $\bmod p$.

For a fixed set $W = \{\overline{w_1}, \dots, \overline{w_s}\}$ corresponding to some term in $T(c_s)$, let f be the injective map from $\{w_1, \dots, w_s\}$ to W defined by

$$f : w_t \to \overline{w_t},$$

and consider S_s the symmetric group of $\{w_1, \dots, w_s\}$. Since a permutation $a \in S_s$ can be written as the product of several separated cycles uniquely, when replace them by their f-values the permutation a determines a table as the form mentioned in **section 3** directly, and each of cycles transforms to a block. Following **lemma 4.2.**, **lemma 4.3.** and **lemma 4.4.** we can easily see that a table is corresponding to a term as (7) if and only if it determines an f-simple permutation of S_s and satisfies the relation (9) in **proposition 3.4.**. Define

$$G = \{a \in S_s \mid \text{the table corresponding to } a \text{ satisfying (9) }\}.$$

Suppose there are k distinct u-values of $\overline{w_t} \in W$, denote by $u_1, \cdots, u_k$. Let

$$W_i = \{w_t \mid \overline{w_t} \in W \text{ and the } u-\text{value of } \overline{w_t} \text{ is equal to } u_i\}$$

and

$$s_i = \mid W_i \mid$$

for $i = 1, \cdots, k$. Then

Lemma 5.1. *Let S_{s_i} be the symmetric group of W_i, then for any $a \in G$, $a^{-1}G = \prod_{i=1}^{k} S_{s_i}$.*

Proof. For any $b \in G$ and any $w \in \{w_1, \dots, w_s\}$, by definition of G, $\overline{w}$'s u-value is equal to $\overline{b(w)}$'s v-value. Thus $\overline{a^{-1}b(w)}$'s u-value is equal to $\overline{w}$'s u-value. This shows that $a^{-1}G \subset \prod_{i=1}^{k} S_{s_i}$.

Besides, for any $c \in \prod_{i=1}^{k} S_{s_i}$ and any $w \in \{w_1, \dots, w_s\}$, $\overline{c(w)}$'s u-value is equal to $\overline{w}$'s u-value. Thus $ac \in G$. This shows that $\prod_{i=1}^{k} S_{s_i} \subset a^{-1}G$.

Then we complete the proof. □

If there are two elements of W having same u-value, then some of $s_i > 1$ and by **lemma 5.1.** the number of even permutations is equal to the number of odd permutations in G.

Since any permutation can be written as a product of several transpositions, it is easily to check that $G_f G = G G_f = G$, thus it satisfies the condition of **proposition 4.5.**.

For any $\sigma \in G_f$ and $a \in G$ an f-simple permutation, $\sigma a \sigma^{-1}$ and a are corresponding to the same term as (7), and so its inverse, i.e. for any two f-simple permutations $a, b \in G$, if they are corresponding to the same term as (7) by replacing f-values, then there exist a $\sigma \in G_f$ such that $b = \sigma a \sigma^{-1}$. Following **proposition 4.6.** we obtain that there is a half number of terms in (8) for which the part $\prod_j(-\prod_{i=1}^m \prod_{t=0}^{s_j-1} \Gamma_p(\{p^t r_{ji}\})) \mod p$ is equal to $\prod_j \prod_{i=1}^m \prod_{t=0}^{s_j-1} \frac{1}{k_{ji}[t]!}$ and the same number terms for which the part $\prod_j(-\prod_{i=1}^m \prod_{t=0}^{s_j-1} \Gamma_p(\{p^t r_{ji}\})) \mod p$ is equal to $-\prod_j \prod_{i=1}^m \prod_{t=0}^{s_j-1} \frac{1}{k_{ji}[t]!}$. By the reason above, we obtain that

Proposition 5.2. *Assume that $p \geq \sum_{i=1}^m d_i$. If there are two vectors $\overline{w}$ of W corresponding to the same value of u, then the sum of all terms as (7) in the same equivalent class has the ord_p-value not smaller than $1 + \frac{1}{p-1}\sum_j \sum_{i=1}^m \sigma((p^{s_j}-1)r_{ji})$.*

Corollary 5.3. *In **proposition 5.2.**, the lower bound can be instead by $1 + \frac{1}{d}\sum_j \sum_{t=0}^{s_j-1} u_j[t]$.*

Proof. By definition, $\sum_{i=1}^m d_i k_{ji}[t] = u_j[t]p - v_j[t]$, thus $\sum_{i=1}^m k_{ji}[t] \geq \frac{1}{d}\sum_{i=1}^m d_i k_{ji}[t] = \frac{1}{d}(u_j[t]p - v_j[t])$. So

$$\begin{aligned}
\frac{1}{p-1}\sum_j \sum_{i=1}^m \sigma((p^{s_j}-1)r_{ji}) &= \frac{1}{p-1}\sum_j \sum_{i=1}^m \sigma(k_{ji}) \\
&= \frac{1}{p-1}\sum_j \sum_{i=1}^m \sum_{t=0}^{s_j-1} k_{ji}[t] \\
&= \frac{1}{p-1}\sum_j \sum_{t=0}^{s_j-1} \sum_{i=1}^m k_{ji}[t] \\
&\geq \frac{1}{p-1}\sum_j \sum_{t=0}^{s_j-1} \frac{1}{d}(u_j[t]p - v_j[t]) \\
&= \frac{1}{d}\sum_j \sum_{t=0}^{s_j-1} u_j[t],
\end{aligned}$$

the last equation follows from the relationship (9). Then we have proved the corollary. □

If $u_j[t] = 0$ for some j and t, then $v_j[t]$ must also be the value 0. By **proposition 3.4.** we can obtain that the values u, v in every column vectors of this block must be 0, it means that this block has only one column. Because any table of term has at most one such a block, thus $\sum_j \sum_{t=0}^{s_j-1} u_j[t] \geq s - 1$, that is

Corollary 5.4. *In **proposition 5.2.**, the lower bound can be instead by $1 + \frac{s-1}{d}$.*

Following **corollary 5.4.** we have proved the first part of **proposition 3.5.**.

By the proof of **corollary 5.3.**, for any $c \in T(c_s)$,

$$\mathrm{ord}_p c \geq \frac{1}{d} \sum_j \sum_{t=0}^{s_j-1} u_j[t].$$

If any two of $u_j[t]$ are all different, then

$$\mathrm{ord}_p c \geq \frac{1}{d} \sum_{u=0}^{s-1} u = \frac{s(s-1)}{2d}.$$

Besides, the Newton polygon of $L(f,T)$ is symmetric in the sense that for every slope segment α there is a slope segment $1-\alpha$ of the same horizontal length, we should only determine half number slope segments of the Newton polygon of $L(f,T)$. In other words, for $L^*(f,T) = (1-T)L(f,T)$, we should only consider those coefficients c_s where s satisfies $s - 1 \leq \frac{d-1}{2}$.

Thus, to prove the last part of **proposition 3.5.**, we should only show that

$$\frac{s(s-1)}{2d} < 1 + \frac{s-1}{d},$$

which is equivalent to $(s-1)(s-2) < 2d$.

6. General calculation of $\mathrm{ord}_p c_s$ for $(s-1)(s-2) < 2d$

Let $m > 1$, $f(x) = \sum_{i=1}^m a_i x^{d_i}$ be a polynomial, where $0 < d_1 < \cdots < d_m = d$ and $a_i \in \mathbf{F}_p^*$, $a_m = 1$. Assume $p \geq \sum_{i=1}^m d_i$. Since such a linear transformation $ax + b$ $(a, b \in \mathbf{F}_q, a \neq 0 \bmod p)$ of x does not change the L-function, we can also assume $d_{m-1} < d - 1$.

By **proposition 3.5.** we should consider such $c \in T(c_s)$ that corresponds to s different u-values beginning from $0, 1, \cdots, s-1$ respectively. Besides, for **proposition 3.4.** the v-values of the table corresponding to c are equal to its u-values respectively. To determine a $c \in T(c_s)$ is equivalent to determine the relevant table. Therefore we should consider these s equations:

$$\sum_{i=1}^m d_i k_{ji}[t] = u_j[t]p - v_j[t] \tag{14}$$

for all j and t such that all $k_{ji}[t]$ are non-negative integers.

Recall that

$$\mathrm{ord}_p c = \frac{1}{p-1} \sum_j \sum_{i=1}^m \sum_{t=0}^{s_j-1} k_{ji}[t],$$

for a series given positive integers $r_j[t]$, we insert s equations

$$\sum_{i=1}^m k_{ji}[t] = r_j[t] \tag{15}$$

into (14). Thus, to calculate $\mathrm{ord}_p c_s$ we should calculate $\mathrm{ord}_p(\sum_{\mathrm{ord}_p c = \frac{r}{p-1}} c)$ satisfying (14) and (15) in order of $r = \sum_j \sum_{t=0}^{s_j-1} r_j[t]$ from small to large.

If some $k_{ji}[t] \geq d$ with $i < m$, then we can use $k_{jm}[t] + d_i$ and $k_{ji}[t] - d$ instead of $k_{jm}[t]$ and $k_{ji}[t]$ satisfying (14) and small $r_j[t] = \sum_{i=1}^{m} k_{ji}[t]$ into $\sum_{i=1}^{m} k_{ji}[t] - d + d_i$. Therefore, to make r smallest all $k_{ji}[t]$ with $i < m$ should be smaller than d.

Combine (14) and (15) we get

$$\sum_{i=1}^{m-1} (d - d_i) k_{ji}[t] = d r_j[t] - u_j[t] p + v_j[t]. \tag{16}$$

Define $C(r; u, v)$ the set of all non-negative integral solutions $[h_1, h_2, \cdots, h_{d-2}]$ of

$$\sum_{i=1}^{m-1} (d - d_i) k_i = dr - up + v$$

with $h_{d_i} = k_i$ for $i = 1, \cdots, m-1$ and $h_j = 0$ for all indexes $j \neq d_1, \cdots, d_m - 1$.

Suppose $u_1, \cdots, u_s$ are s distinct non-negative integers, and $\sigma \in S_s$ is a permutation on $\{1, \cdots, s\}$. Let $r = \sum_{i=1}^{s} r_i$. When we select s solutions in $C(r_1; u_1, u_{\sigma(1)}), \cdots,$ $C(r_s; u_s, u_{\sigma(s)})$ respectively, by **proposition 3.4.** they construct a table uniquely, which is corresponding to a term $c \in T(c_s)$ with $\mathrm{ord}_p c = \frac{r}{p-1}$.

Recall **section 5**, the part

$$\prod_j \Big(- \prod_{i=1}^{m} \prod_{t=0}^{s_j - 1} \Gamma_p(\{p^t r_{ji}\})\Big)$$

mod p is equal to $\prod_j \prod_{i=1}^{m} \prod_{t=0}^{s_j-1} \frac{1}{k_{ji}[t]!}$ or its inverse, it depends on whether the number of blocks in the corresponding table is even or not, i.e. it depends on whether the permutation

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

is even or odd. We can also calculate $\prod_j \prod_{i=1}^{m} \prod_{t=0}^{s_j-1} \frac{1}{k_{ji}[t]!}$ column by column in the table, i.e. a column $[k_1, \cdots, k_m]^T = [h_{d_1}, \cdots, h_{d_{m-1}}, r - \sum_{i=1}^{m-1} h_{d_i}]^T$ with $[h_1, \cdots, h_{d-2}] \in C(r; u, v)$ supplies the part

$$\Big(\prod_{i=1}^{m-1} \frac{1}{k_i!}\Big) \cdot \frac{1}{(r - \sum_{i=1}^{m-1} k_i)!} \bmod p.$$

Define

$$F(r; u, v) = \sum_{[h_1, \cdots, h_{d-2}] \in C(r; u, v)} \Big(\prod_{i=1}^{m-1} \frac{1}{k_i!} \chi(a_i)^{k_i}\Big) \cdot \frac{1}{(r - \sum_{i=1}^{m-1} k_i)!}$$

and

$$F_r^s = \sum_{\sigma \in S_s} \sum_{\sum_{i=1}^{s} r_i = r} \sum_{u_1, \cdots, u_s \text{ are all distinct}} \mathrm{sign}(\sigma) \prod_{i=1}^{s} F(r_i; u_i, u_{\sigma(i)}).$$

If we define $O(c_s) = \{c \in T(c_s) \mid \text{all the } u\text{-values corresponding to } c \text{ are distinct each other.}\}$, then we have

$$\pi^{-r} \Big(\sum_{c \in O(c_s),\ \mathrm{ord}_p c = \frac{r}{p-1}} c \Big) \equiv F_r^s \bmod p.$$

Note that when r is small, the $\sum_{u_1,\cdots,u_s \text{ are all distinct}}$ of F_r^s only contains the case $\{u_1,\cdots,u_s\}=\{0,\cdots,s-1\}$.

By **proposition 3.5.**, to determine $\mathrm{ord}_p c_s$, we should only begin to calculate F_r^s in order of r from small to large until for the first r satisfying $F_r^s \neq 0 \bmod p$. we also see that all r_i are disjointed, then to make r smallest we should only make each r_i smallest.

By **proposition 3.5.** again we have

Theorem 6.1. *Suppose $F_r^s = 0$ when $r < R$ and $F_R^s \neq 0$, and denote $\lambda_s = \frac{R}{p-1}$. If $\lambda_s < 1 + \frac{s-1}{d}$ then*

$$ord_p c_s = \lambda_s.$$

Remark 6.2. *This theorem give a feasible method to calculate Newton polygon of L-function in sense of Sperber's theorem on case $d = 3$ (see [16]).*

7. General calculation for $d = 3, 4, 6$

Let $\{\omega_0, \ldots, \omega_{d-1}\}$ be the set of reciprocal roots of $L^*(f,T)$ satisfying

$$0 = \mathrm{ord}_p\omega_0 \leq \cdots \leq \mathrm{ord}_p\omega_{d-1}$$

then $\{\omega_1, \ldots, \omega_{d-1}\}$ is the set of reciprocal roots of $L(f,T)$.

If $p \equiv 1 \bmod d$, then $[0,0,\cdots,0] \in C(\frac{u(p-1)}{d}; u, u)$ $(u = 0, 1, \cdots, s-1)$ is the unique solution satisfying (16) that achieves the lower bound $\sum_{u=0}^{s-1} \frac{u(p-1)}{d}$ of $r = \sum_{i=1}^{s} r$. Thus

$$\mathrm{ord}_p\omega_s = \frac{s-1}{d}$$

for all s. This is the 1-dimension case of A.S. conjecture.

Generally, by the discussion above, we first consider the terms in c_s of which W-set is corresponding to the u-values set as $\{0, 1, \ldots, s-1\}$ and calculate the sum of terms among them which has the smallest ord_p-value. if the sum not increase the ord_p-value, then it is the ord_p-value of c_s; otherwise the sum of their Γ_p-part is equal to 0 mod p, therefore the ord_p-value at least increase 1 and then we calculate the sum of terms among them having the next bigger ord_p-value and so on. Note that $\mathrm{ord}_p c_1$ is always equal to 0, we calculate $\mathrm{ord}_p c_2$ first of all, it is corresponding to the first slope segment of Newton polygon of $L(f,T)$.

Since the Newton polygon of $L(f,T)$ is symmetric in the sense that for every slope segment α there is a slope segment $1-\alpha$ of the same horizontal length, $x - 1 = 2y$ lies upon the Newton polygon. Besides, the points $(s, 1 + \frac{s-1}{d})$ are all on the line $y = 1 + \frac{x-1}{d}$. Thus if $(1 + \frac{d-1}{2}, 1 + \frac{d-1}{2d})$ is over line $x - 1 = 2y$, i.e. if $d \leq 6$, then following **proposition 3.5.** we have determined the Newton polygon by **theorem 6.1.**.

Theorem 7.1. *Let $f(x) = x^3 + a_1x$, $a_1 \neq 0$ and $p > 3$, $p \equiv 2$ mod 3. Thus*

$$ord_p\omega_1 = \frac{p+1}{3(p-1)}, \; ord_p\omega_2 = 1 - \frac{p+1}{3(p-1)}.$$

Proof. Let $s = 2$, then we should consider (16) when $u = v = 1$. Since $\frac{p+1}{3}$ is an integer, thus $r = \frac{p+1}{3}$ is the lower bound of r while $C(\frac{p+1}{3}; 1, 1) = \{[1]\}$. So $F(\frac{p+1}{3}; 1, 1) = \frac{1}{(\frac{p-2}{3})!}\chi(a_1)$ and $F^2_{\frac{p+1}{3}} = \frac{1}{(\frac{p-2}{3})!}\chi(a_1) \neq 0 \bmod p$. Thus $\mathrm{ord}_p c_2 = \lambda_2 = \frac{p+1}{3(p-1)}$ and we finish the proof. $\square$

Remark 7.2. *Assume $f(x) = x^3 + a_2x^2 + a_1x$, $a_1 \neq 0, a_2 \neq 0$, $p > 3$, $p \equiv 2 \ mod\ 3$. Then the lower bound of r is $\frac{p+1}{3}$ and $C(\frac{p+1}{3}; 1, 1) = \{[1, 0], [0, 2]\}$. So*

$$\begin{aligned} F^2_{\frac{p+1}{3}} &= \frac{1}{(\frac{p-2}{3})!}\chi(a_1) + \frac{1}{2!} \cdot \frac{1}{(\frac{p-2}{3} - 1)!}\chi(a_2)^2 \\ &\equiv \frac{1}{3(\frac{p-2}{3})!}(3\chi(a_1) - \chi(a_2)^2) \ mod\ p. \end{aligned}$$

If $F^2_{\frac{p+1}{3}} \equiv 0 \ mod\ p$, i.e. $3\chi(a_1) - \chi(a_2)^2 = 0 \ mod\ p$, we will consider the second smallest value of r, probably is $\frac{p+1}{3} + 1$. For $C(\frac{p+1}{3} + 1; 1, 1) = \{[2, 1], [1, 3], [0, 5]\}$,

$$F^2_{\frac{p+1}{3}+1} = \chi(a_1)^2\chi(a_2)\cdot\frac{1}{2!}\cdot\frac{1}{(\frac{p-2}{3} - 1)!} + \chi(a_1)\chi(a_2)^3\cdot\frac{1}{3!}\cdot\frac{1}{(\frac{p-2}{3} - 2)!} + \chi(a_2)^5\cdot\frac{1}{5!}\cdot\frac{1}{(\frac{p-2}{3} - 3)!}.$$

Since $3\chi(a_1) - \chi(a_2)^2 = 0 \ mod\ p$, we have

$$\begin{aligned} F^2_{\frac{p+1}{3}+1} &\equiv \chi(a_2)^5 \cdot (\frac{1}{3^2} \cdot \frac{1}{2!} \cdot \frac{1}{(\frac{p-2}{3} - 1)!} + \frac{1}{3} \cdot \frac{1}{3!} \cdot \frac{1}{(\frac{p-2}{3} - 2)!} + \frac{1}{5!} \cdot \frac{1}{(\frac{p-2}{3} - 3)!}) \\ &\equiv \frac{\chi(a_2)^5}{(\frac{p-2}{3})!} \cdot (\frac{1}{3^2} \cdot \frac{1}{2!} \cdot \frac{-2}{3} + \frac{1}{3} \cdot \frac{1}{3!} \cdot (\frac{-2}{3})(\frac{-2}{3} - 1) + \frac{1}{5!} \cdot (\frac{-2}{3})(\frac{-2}{3} - 1)(\frac{-2}{3} - 2)) \\ &\equiv 0 \ mod\ p, \end{aligned}$$

and then we will consider the third smallest value of r, and so on. This is similar as [16].

However, if we changing x by $x - \frac{1}{3}a_2$, following $3\chi(a_1) - \chi(a_2)^2 = 0 \ mod\ p$ we have

$$f(x - \frac{1}{3}a_2) = x^3 - \frac{1}{27}a_2^3,$$

it is diagonal case, so that

$$ord_p\omega_1 = ord_p\omega_2 = \frac{1}{2}.$$

It is better than [16].

Theorem 7.3. *Let $f(x) = x^4 + a_2x^2 + a_1x$ and $p > 6$, $p \equiv 3 \ mod\ 4$.*

If $a_2 \neq 0$, then

$$ord_p\omega_1 = \frac{p+1}{4(p-1)}, ord_p\omega_2 = \frac{1}{2}, ord_p\omega_3 = 1 - \frac{p+1}{4(p-1)};$$

If $a_2 = 0$ and $a_1 \neq 0$, then

$$ord_p\omega_1 = \frac{p+5}{4(p-1)}, ord_p\omega_2 = \frac{1}{2}, ord_p\omega_3 = 1 - \frac{p+5}{4(p-1)}.$$

Proof. Let $s = 2$, then we should consider (16) when $u = v = 1$.

Suppose $a_2 \neq 0$, then $r = \frac{p+1}{4}$ is the lower bound of r while $C(\frac{p+1}{4}; 1, 1) = \{[0, 1]\}$. So $F(\frac{p+1}{4}; 1, 1) = \frac{1}{(\frac{p-3}{4})!}\chi(a_2)$ and $F^2_{\frac{p+1}{4}} = \frac{1}{(\frac{p-3}{4})!}\chi(a_2) \neq 0 \bmod p$. Thus $\mathrm{ord}_p c_2 = \lambda_2 = \frac{p+1}{4(p-1)}$. We finish the proof of the first case.

Suppose $a_2 = 0, a_1 \neq 0$, then $r = \frac{p+5}{4}$ is the lower bound of r while $C(\frac{p+5}{4}; 1, 1) = \{[2, 0]\}$. So $F(\frac{p+5}{4}; 1, 1) = \frac{1}{2!} \cdot \frac{1}{(\frac{p-3}{4})!}\chi(a_1)^2$ and $F^2_{\frac{p+5}{4}} = \frac{1}{2!} \cdot \frac{1}{(\frac{p-3}{4})!}\chi(a_1)^2 \neq 0 \bmod p$. Thus $\mathrm{ord}_p c_2 = \lambda_2 = \frac{p+5}{4(p-1)}$. We finish the proof of the last case. □

Remark 7.4. *In fact the first part in this theorem has two cases: $a_1 = 0$ and $a_1 \neq 0$. Since $C(\frac{p+1}{4}; 1, 1) = \{[0, 1]\}$ does not depend on whether $a_1 = 0$, it does not affect the calculation of $F^2_{\frac{p+5}{4}}$. That is why we defined $C(r; u, v)$ like that.*

To simply describe the results, we use a_i directly to instead $\chi(a_i)$.

Theorem 7.5. *Let $f(x) = x^6 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$ and $p \geq 6 + \sum_{a_i \neq 0} i$, $p \equiv -1 \bmod 6$.*

(i) If $a_4 \neq 0$, $3a_3^2 + a_4^3 - 3a_2a_4 \neq 0$, then

$$ord_p\omega_1 = \frac{p+1}{6(p-1)}, ord_p\omega_2 = \frac{p+1}{3(p-1)}, ord_p\omega_3 = \frac{1}{2}, ord_p\omega_4 = 1 - \frac{p+1}{3(p-1)},$$
$$ord_p\omega_5 = 1 - \frac{p+1}{6(p-1)}.$$

(ii) If $p \geq 17$ (resp. $p = 11$), $a_4 \neq 0$, $3a_3^2 + a_4^3 - 3a_2a_4 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 \neq 0$ (resp. $-4a_1^2a_4^4 + 2a_3^2a_4^6 - a_3^4a_4^3 - 4a_3^6 + a_1a_3a_4^5 - 3a_1a_3^3a_4^2 \neq 0$), then

$$ord_p\omega_1 = \frac{p+1}{6(p-1)}, ord_p\omega_2 = \frac{p+4}{3(p-1)}, ord_p\omega_3 = \frac{1}{2}, ord_p\omega_4 = 1 - \frac{p+4}{3(p-1)},$$
$$ord_p\omega_5 = 1 - \frac{p+1}{6(p-1)}.$$

(iii) If $p \geq 29$ (resp. $p = 11$, $p = 17$, $p = 23$), $a_4 \neq 0$, $3a_3^2 + a_4^3 - 3a_2a_4 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$ (resp. $-4a_1^2a_4^4 + 2a_3^2a_4^6 - a_3^4a_4^3 - 4a_3^6 + a_1a_3a_4^5 - 3a_1a_3^3a_4^2 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$), $48384a_3^8 + 225a_4^{12} + 5600a_4^9a_3^2 + 41888a_4^6a_3^4 + 80640a_4^3a_3^6 \neq 0$ (resp. $5 + 5a_3^4a_4^4 + 2a_3^6a_4 - 2a_3^8a_4^8 - 2a_4^5 + 5a_3^2a_4^2 - 2a_3^4a_4^9 - 2a_3^6a_4^6 + 2a_3^8a_4^3 \neq 0$, $-4a_3^8 + 8a_4^5a_3^4 + 7a_3^5a_4^4 - 2a_3^7a_4^2 \neq 0$, $-10a_3^8 + 11a_4^{12} + 8a_3^2a_4^9 - 11a_3^4a_4^6 - 9a_3^6a_4^3 \neq 0$), then

$$ord_p\omega_1 = \frac{p+1}{6(p-1)}, ord_p\omega_2 = \frac{p+7}{3(p-1)} (when\ p = 11\ here\ is\ \frac{1}{2}\ instead),$$
$$ord_p\omega_3 = \frac{1}{2}, ord_p\omega_4 = 1 - \frac{p+7}{3(p-1)} (when\ p = 11\ here\ is\ \frac{1}{2}\ instead),$$
$$ord_p\omega_5 = 1 - \frac{p+1}{6(p-1)}.$$

(iv) If $p \geq 29$ (resp. $p = 17$, $p = 23$), $a_4 \neq 0$, $3a_3^2 + a_4^3 - 3a_2a_4 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$ (resp. $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$, $7a_3a_4^3 + 12a_3^3 - 12a_1a_4^2 = 0$),

$48384a_3^8+225a_4^{12}+5600a_4^9a_3^2+41888a_4^6a_3^4+80640a_4^3a_3^6=0$ (*resp.* $-4a_3^8+8a_4^5a_3^4+7a_3^5a_4^4-2a_3^7a_4^2=0$, $-10a_3^8+11a_4^{12}+8a_3^2a_4^9-11a_3^4a_4^6-9a_3^6a_4^3=0$), *then*

$$ord_p\omega_1=\frac{p+1}{6(p-1)}, ord_p\omega_2=\frac{p+10}{3(p-1)}(\textit{when } p=17 \textit{ here is } \frac{1}{2} \textit{ instead}),$$
$$ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p+10}{3(p-1)}(\textit{when } p=17 \textit{ here is } \frac{1}{2} \textit{ instead}),$$
$$ord_p\omega_5=1-\frac{p+1}{6(p-1)}.$$

(v) If $a_3=a_1=0$, $3a_2=a_4^2\neq 0$, *then*

$$ord_p\omega_1=\frac{p+1}{6(p-1)}, ord_p\omega_2=ord_p\omega_3=ord_p\omega_4=\frac{1}{2}, ord_p\omega_5=1-\frac{p+1}{6(p-1)}.$$

(vi) If $a_4=0$, $a_3\neq 0$, $a_2^2+2a_1a_3\neq 0$, *and* $a_1\neq 0$ *or* $a_2\neq 0$, *then*

$$ord_p\omega_1=\frac{p+7}{6(p-1)}, ord_p\omega_2=\frac{p-2}{3(p-1)}, ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p-2}{3(p-1)},$$
$$ord_p\omega_5=1-\frac{p+7}{6(p-1)}.$$

(vii) If $a_4=a_3=0$, $a_2\neq 0$, *then*

$$ord_p\omega_1=\frac{p+7}{6(p-1)}, ord_p\omega_2=\frac{p+1}{3(p-1)}, ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p+1}{3(p-1)},$$
$$ord_p\omega_5=1-\frac{p+7}{6(p-1)}.$$

(viii) If $a_4=0$, $a_3\neq 0$, $a_2^2+2a_1a_3=0$, $9a_2^3-10a_3^4\neq 0$, *and* $a_1\neq 0, a_2\neq 0$, *then*

$$ord_p\omega_1=\frac{p+13}{6(p-1)}, ord_p\omega_2=\frac{p-5}{3(p-1)}, ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p-5}{3(p-1)},$$
$$ord_p\omega_5=1-\frac{p+13}{6(p-1)}.$$

(ix) If $a_4=0$, $a_3\neq 0$, $a_2^2+2a_1a_3=0$, $9a_2^3-10a_3^4=0$, *then*

$$ord_p\omega_1=\frac{p+19}{6(p-1)}, ord_p\omega_2=\frac{p-8}{3(p-1)}, ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p-8}{3(p-1)},$$
$$ord_p\omega_5=1-\frac{p+19}{6(p-1)}.$$

(x) If $a_4=a_3=a_2=0$, $a_1\neq 0$, *then*

$$ord_p\omega_1=\frac{p+19}{6(p-1)}, ord_p\omega_2=\frac{p+4}{3(p-1)}, ord_p\omega_3=\frac{1}{2}, ord_p\omega_4=1-\frac{p+4}{3(p-1)},$$
$$ord_p\omega_5=1-\frac{p+19}{6(p-1)}.$$

(xi) If $a_4 = a_2 = a_1 = 0$, $a_3 \neq 0$, then

$$ord_p\omega_1 = ord_p\omega_2 = \frac{p+1}{4(p-1)}, ord_p\omega_3 = \frac{1}{2}, ord_p\omega_4 = ord_p\omega_5 = 1 - \frac{p+1}{4(p-1)}.$$

Proof. Similar as above, we can generally obtain that

$$C(\frac{p+1}{6};1,1) = \{[0,0,0,1]\},$$

$$C(1+\frac{p+1}{6};1,1) = \{[0,0,0,4],[0,1,0,2],[0,2,0,0],[1,0,1,0],[0,0,2,1]\},$$

$$\begin{aligned} C(2+\frac{p+1}{6};1,1) &= \{[0,0,0,7],[2,0,0,2],[0,1,0,5],[2,1,0,0],[0,2,0,3],[0,3,0,1], \\ &\quad [1,0,1,3],[1,1,1,1],[0,0,2,4],[0,1,2,2],[0,2,2,0],[1,0,3,0], \\ &\quad [0,0,4,1]\}, \end{aligned}$$

$$\begin{aligned} C(3+\frac{p+1}{6};1,1) &= \{[0,0,0,10],[2,0,0,5],[4,0,0,0],[0,1,0,8],[2,1,0,3],[0,2,0,6], \\ &\quad [2,2,0,1],[0,3,0,4],[0,4,0,2],[0,5,0,0],[1,0,1,6],[3,0,1,1], \\ &\quad [1,1,1,4],[1,2,1,2],[1,3,1,0],[0,0,2,7],[2,0,2,2],[0,1,2,5], \\ &\quad [2,1,2,0],[0,2,2,3],[0,3,2,1],[1,0,3,3],[1,1,3,1],[0,0,4,4], \\ &\quad [0,1,4,2],[0,2,4,0],[1,0,5,0],[0,0,6,1]\}, \end{aligned}$$

and

$$C(\frac{p+1}{3};2,2) = \{[0,0,0,2],[0,1,0,0]\},$$

$$\begin{aligned} C(1+\frac{p+1}{3};2,2) &= \{[0,0,0,5],[2,0,0,0],[0,1,0,3],[0,2,0,1],[1,0,1,1],[0,0,2,2], \\ &\quad [0,1,2,0]\}, \end{aligned}$$

$$\begin{aligned} C(2+\frac{p+1}{3};2,2) &= \{[0,0,0,8],[2,0,0,3],[0,1,0,6],[2,1,0,1],[0,2,0,4],[0,3,0,2], \\ &\quad [0,4,0,0],[1,0,1,4],[1,1,1,2],[1,2,1,0],[0,0,2,5],[2,0,2,0], \\ &\quad [0,1,2,3],[0,2,2,1],[1,0,3,1],[0,0,4,2],[0,1,4,0]\}, \end{aligned}$$

$$
\begin{aligned}
C(3+\frac{p+1}{3};2,2) &= \{[0,0,0,11],[2,0,0,6],[4,0,0,1],[0,1,0,9],[2,1,0,4],[0,2,0,7],\\
&[2,2,0,2],[0,3,0,5],[2,3,0,0],[0,4,0,3],[0,5,0,1],[1,0,1,7],\\
&[3,0,1,2],[1,1,1,5],[3,1,1,0],[1,2,1,3],[1,3,1,1],[0,0,2,8],\\
&[2,0,2,3],[0,1,2,6],[2,1,2,1],[0,2,2,4],[0,3,2,2],[0,4,2,0],\\
&[1,0,3,4],[1,1,3,2],[1,2,3,0],[0,0,4,5],[2,0,4,0],[0,1,4,3],\\
&[0,2,4,1],[1,0,5,1],[0,0,6,2],[0,1,6,0]\},
\end{aligned}
$$

and

$$C(\frac{p+1}{6};1,2) = \{[0,0,1,0]\},$$

$$C(1+\frac{p+1}{6};1,2) = \{[1,0,0,2],[1,1,0,0],[0,0,1,3],[0,1,1,1],[0,0,3,0]\},$$

$$
\begin{aligned}
C(2+\frac{p+1}{6};1,2) &= \{[1,0,0,5],[3,0,0,0],[1,1,0,3],[1,2,0,1],[0,0,1,6],[2,0,1,1],\\
&[0,1,1,4],[0,2,1,2],[0,3,1,0],[1,0,2,2],[1,1,2,0],[0,0,3,3],\\
&[0,1,3,1],[0,0,5,0]\},
\end{aligned}
$$

$$
\begin{aligned}
C(3+\frac{p+1}{6};1,2) &= \{[1,0,0,8],[3,0,0,3],[1,1,0,6],[3,1,0,1],[1,2,0,4],[1,3,0,2],\\
&[1,4,0,0],[0,0,1,9],[2,0,1,4],[0,1,1,7],[2,1,1,2],[0,2,1,5],\\
&[2,2,1,0],[0,3,1,3],[0,4,1,1],[1,0,2,5],[3,0,2,0],[1,1,2,3],\\
&[1,2,2,1],[0,0,3,6],[2,0,3,1],[0,1,3,4],[0,2,3,2],[0,3,3,0],\\
&[1,0,4,2],[1,1,4,0],[0,0,5,3],[0,1,5,1],[0,0,7,0]\},
\end{aligned}
$$

and

$$C(\frac{p+1}{3};2,1) = \{[0,0,1,0]\},$$

$$C(1+\frac{p+1}{3};2,1) = \{[1,0,0,2],[1,1,0,0],[0,0,1,3],[0,1,1,1],[0,0,3,0]\},$$

$$
\begin{aligned}
C(2+\frac{p+1}{3};2,1) &= \{[1,0,0,5],[3,0,0,0],[1,1,0,3],[1,2,0,1],[0,0,1,6],[2,0,1,1],\\
&[0,1,1,4],[0,2,1,2],[0,3,1,0],[1,0,2,2],[1,1,2,0],[0,0,3,3],\\
&[0,1,3,1],[0,0,5,0]\},
\end{aligned}
$$

$$
\begin{aligned}
C(3+\frac{p+1}{3};2,1) \;=\; & \{[1,0,0,8],[3,0,0,3],[1,1,0,6],[3,1,0,1],[1,2,0,4],[1,3,0,2],\\
& [1,4,0,0],[0,0,1,9],[2,0,1,4],[0,1,1,7],[2,1,1,2],[0,2,1,5],\\
& [2,2,1,0],[0,3,1,3],[0,4,1,1],[1,0,2,5],[3,0,2,0],[1,1,2,3],\\
& [1,2,2,1],[0,0,3,6],[2,0,3,1],[0,1,3,4],[0,2,3,2],[0,3,3,0],\\
& [1,0,4,2],[1,1,4,0],[0,0,5,3],[0,1,5,1],[0,0,7,0]\}.
\end{aligned}
$$

Thus we can calculate every F_r^s $(s=1,2)$ and get conclusions by discussion on whether $F_r^s=0$ for some r and s. $\square$

Remark 7.6. *When $p\le 29$, some $C(r;u,v)$ in the proof will lost some elements. For example, when $p=23$, $C(2+\frac{p+1}{6};1,1)$ will lost $[0,0,0,7]$ and $C(3+\frac{p+1}{6};1,1)$ will lost $[0,0,0,10],[0,1,0,8],[0,2,0,6],[1,0,1,6],[0,0,2,7],[0,0,4,4]$. The reason is, $r-\sum_{i=1}^{m-1}k_i$ must be a non-negative integer. So these cases when p are small should be considered specially, see case (ii) and (iii).*

*Most cases in **theorem 7.5.** are consistent with S.Hong's result (see [10]), but some cases did not discuss by S.Hong. For example, S.Hong lost the case (ii), it is possible since $a_1=2, a_2=6, a_3=a_4=3$ is such an example.*

Furthermore the first case in S.Hong's Theory ($a_1a_2a_3a_4\neq 0$) had some mistake for the same reason.

8. **Some other examples**

To illustrate our method, consider the case where $f=x^7+ax^4$, $a\in\mathbf{F}_p^*$ and $p=5$ a small prime. Similar to the method in **section 3**, we also have a table to indicate the factor in (7), the only different is, it do not need satisfy the relation (9).

It is clear that $L^*(f/\mathbf{F}_p,T)$ is a polynomial of degree 7. We need to consider the solutions $r=(r_1,r_2)\in S_p(f)$ of equation

$$
\begin{pmatrix} 4,7 \end{pmatrix}\begin{pmatrix} r_1\\ r_2\end{pmatrix}\equiv 0 \bmod 1. \tag{17}
$$

Consider the coefficient c_2, $\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ is the only table which satisfies (17) and makes the sum of all elements in the table smallest. So we have

$$
\mathrm{ord}_p c_2=\frac{1}{4}.
$$

Consider the coefficient c_3, $\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}$ is the only table which satisfies (17) and makes the sum of all elements in the table smallest. So we have

$$
\mathrm{ord}_p c_3=\frac{3}{4}.
$$

Consider the coefficient c_4, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1,0 \\ 0,4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0,1,1 \\ 1,1,2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0,2,0 \\ 0,2,2 \end{pmatrix}$ are the only 4 tables which satisfies (17) and makes the sum of all elements in the table smallest respectively. Then the sum of their Γ_p-parts $\mod p$ is equal to

$$\frac{1}{2!}\cdot\frac{1}{3!}\cdot\chi(a)^{1+2+3}-\frac{1}{4!}\cdot\chi(a)^{1+1}+\frac{1}{2!}\cdot\chi(a)^{1+1}+\frac{1}{2!}\cdot\frac{1}{2!}\cdot\frac{1}{2!}\cdot\chi(a)^2,$$

that is

$$\frac{1}{12}\chi(a)^2(\chi(a)^4+7).$$

Since $a\in\mathbf{F}_5^*$, we have $\chi(a)^4=1$ and therefore $\frac{1}{12}\chi(a)^2(\chi(a)^4+7)\neq 0 \mod 5$. That means

$$\mathrm{ord}_p c_4=\frac{3}{2}.$$

For the reason that the Newton polygon of $L(f,T)$ is symmetric in the sense that for every slope segment α there is a slope segment $1-\alpha$ of the same horizontal length, and $\mathrm{ord}_p c_3-\mathrm{ord}_p c_2=\frac{1}{2}$, $\mathrm{ord}_p c_4-\mathrm{ord}_p c_3=\frac{3}{4}>\frac{1}{2}$, we have

$$\mathrm{ord}_p\omega_1=\frac{1}{4},$$

$$\mathrm{ord}_p\omega_2=\mathrm{ord}_p\omega_3=\mathrm{ord}_p\omega_4=\mathrm{ord}_p\omega_5=\frac{1}{2},$$

$$\mathrm{ord}_p\omega_6=\frac{3}{4}.$$

An other example is for $f=x^3+axy+by^2$, where $q=p^k$ and $a,b\in\mathbf{F}_q^*$ and $p>6$ is prime satisfying $p\equiv -1 \mod 3$.

It is clear that $L^*(f/\mathbf{F}_q,T)^{(-1)}$ is a polynomial of degree 6. We need to consider the solutions $r=(r_1,r_2,r_3)\in S_p(f)$ of equation

$$\begin{pmatrix} 3,1,0 \\ 0,1,2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}\equiv 0 \mod 1. \tag{18}$$

Consider the coefficient c_2, the solutions $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ of (18) are made up of a table below:

$$\begin{pmatrix} 0,\cdots,0 \\ 0,\cdots,0 \\ 0,\cdots,0 \end{pmatrix}\begin{pmatrix} 0,\cdots,0 \\ 0,\cdots,0 \\ \frac{p-1}{2},\cdots,\frac{p-1}{2} \end{pmatrix}$$

Each block in this table has k columns. It is the only table which satisfies (18) and makes the sum of all elements in it smallest. So we have

$$\mathrm{ord}_q c_2 = \frac{1}{2}.$$

Consider the coefficient c_3, the solutions $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix}$ of (18) are made up of a table below:

(i) $k \equiv 1 \bmod 2$, then

$$\begin{pmatrix} 0, \cdots, 0 \\ 0, \cdots, 0 \\ 0, \cdots, 0 \end{pmatrix} \begin{pmatrix} \frac{2p-1}{3}, & \frac{p-2}{3}, & \cdots, & \frac{2p-1}{3}, & \frac{p-2}{3} \\ 0, & 0, & \cdots, & 0, & 0 \\ 0, & 0, & \cdots, & 0, & 0 \end{pmatrix}$$

The first block in this table has k columns and the second block has $2k$ columns.

(ii) $k \equiv 0 \bmod 2$, then

$$\begin{pmatrix} 0, \cdots, 0 \\ 0, \cdots, 0 \\ 0, \cdots, 0 \end{pmatrix} \begin{pmatrix} \frac{p-2}{3}, & \frac{2p-1}{3}, & \cdots, & \frac{p-2}{3}, & \frac{2p-1}{3} \\ 0, & 0, & \cdots, & 0, & 0 \\ 0, & 0, & \cdots, & 0, & 0 \end{pmatrix} \begin{pmatrix} \frac{2p-1}{3}, & \frac{p-2}{3}, & \cdots, & \frac{2p-1}{3}, & \frac{p-2}{3} \\ 0, & 0, & \cdots, & 0, & 0 \\ 0, & 0, & \cdots, & 0, & 0 \end{pmatrix}$$

Each block in this table has k columns.

It is the only table which satisfies (18) and makes the sum of all elements in it smallest. So we have

$$\mathrm{ord}_q c_3 = 1.$$

Consider the coefficient c_4, we can easily prove that add $\begin{pmatrix} 0, \cdots, 0 \\ 0, \cdots, 0 \\ \frac{p-1}{2}, \cdots, \frac{p-1}{2} \end{pmatrix}$ to the table (i) or (ii) is the only table which satisfies (18) and makes the sum of all elements in it smallest. So we have

$$\mathrm{ord}_q c_4 = \frac{3}{2}.$$

Consider the coefficient c_5 and note the solution $\begin{pmatrix} \frac{p-5}{3(p-1)} \\ \frac{4}{p-1} \\ \frac{p-5}{2(p-1)} \end{pmatrix}$ of (18). We can easily prove that add $\begin{pmatrix} \frac{p-5}{3}, \cdots, \frac{p-5}{3} \\ 4, \cdots, 4 \\ \frac{p-5}{2}, \cdots, \frac{p-5}{2} \end{pmatrix}$ to the table of case c_4 is the only table which satisfies (18) and makes the sum of all elements in it smallest. So we have

$$\mathrm{ord}_q c_5 = \frac{7}{3} + \frac{2}{3(p-1)}.$$

We can not calculate $\mathrm{ord}_q c_6$ by our method yet but $\mathrm{ord}_p c_6$ since

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{p-1}{2} \end{pmatrix} \begin{pmatrix} \frac{2p-1}{3}, \frac{p-2}{3} \\ 0, 0 \\ 0, 0 \end{pmatrix} \begin{pmatrix} \frac{p-5}{3} \\ 4 \\ \frac{p-5}{2} \end{pmatrix} \begin{pmatrix} \frac{2p-4}{3} \\ 2 \\ \frac{p-3}{2} \end{pmatrix}$$

is the only table which satisfies (18) and makes the sum of all elements in it smallest. So we have

$$\mathrm{ord}_p c_6 = \frac{7}{2} + \frac{1}{p-1}.$$

Then for $\frac{1}{1-T} \cdot L^*(f/\mathbf{F}_q, T)^{(-1)}$ we have

$$\mathrm{ord}_q\omega_1 = \mathrm{ord}_q\omega_2 = \mathrm{ord}_p\omega_3 = \frac{1}{2},$$

$$\mathrm{ord}_q\omega_4 = \frac{5}{6} + \frac{2}{3(p-1)}.$$

and for $q = p$ we have

$$\mathrm{ord}_p\omega_5 = \frac{7}{6} + \frac{1}{3(p-1)}.$$

Remark 8.1. *our calculations here are based on* ***theorem 2.2.*** *and* ***section 3****. We can also see that the table-representation and* ***proposition 3.4.*** *are applicable not only for $q = p$ or $n = 1$.*

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